

Some results on perturbations to Lyapunov exponents

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Abstract

In this paper, we study two properties of the Lyapunov exponents under small perturbations: one is when we can remove zero Lyapunov exponents and the other is when we can distinguish all the Lyapunov exponents. The first result shows that we can perturb all the zero integrated Lyapunov exponents $\int_M \lambda_j(x) d\omega(x)$ into nonzero ones, for any partially hyperbolic diffeomorphism. The second part contains an example which shows the local genericity of diffeomorphisms with non-simple spectrum and three results: one discusses the relation between simple-spectrum property and the existence of complex eigenvalues; the other two describe the difference on the spectrum between the diffeomorphisms far from homoclinic tangencies and those in the interior of the complement. Moreover, among the conservative diffeomorphisms far from tangencies, we prove that ergodic ones form a residual subset.

1 Introduction

It was shown in the 1960s that uniformly hyperbolic systems are not dense in the space of dynamical systems. This brought about the naissance of the notions of partial and nonuniform hyperbolicity. Using the concept of Lyapunov exponents, Pesin theory of nonuniformly hyperbolic systems (characterized by all the Lyapunov exponents are non-null for some invariant measure) gives us a rich information about geometric properties of the system. In particular, the points with all Lyapunov exponents non-zero have well-defined unstable and stable invariant manifolds. These tools are the base of most of the results on dynamical systems nowadays. Thus it is of utmost importance to detect when the zero exponents can be removed by perturbations.

One central result in this direction for discrete system is the Shub-Wilkinson example[32]. They build a conservative perturbation to a skew product of an Anosov diffeomorphism of the torus \mathcal{T}^2 by rotations and creates positive exponents in the center direction for Lebesgue almost every point. Baraviera-Bonatti present a local version of Shub-Wilkinson's argument, allowing one to perturb the sum of all the center integrated Lyapunov exponents of any conservative partially hyperbolic systems.

Highly inspired by their results we prove in the present paper that one can perturb every integrated Lyapunov exponent $\int_M \lambda_j(x) d\omega(x)$ of any conservative partially hyperbolic systems to nonzero ones, not only the **sum** of them. This is a generalization to Baraviera-Bonatti[8].

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Let M be a d -dimensional compact Riemannian manifold without boundary ($d \geq 2$) and ω be a smooth volume form. Denote by $\text{Diff}_\omega^1(M)$ the set of volume preserving C^1 -diffeomorphisms on M . Take $f \in \text{Diff}_\omega^1(M)$. By Oseledec Theorem, there exists a Df -invariant decomposition

$$T_x M = \bigoplus_{i=1}^{s(f,x)} E^i(x) \quad \omega - a.e. x \in M \quad (1.1)$$

such that

$$\lim_{m \rightarrow \pm\infty} \frac{1}{m} \log \|d_x f^m \nu\| = \lambda_i(f, x)$$

converges uniformly on $\{\nu \in E^i(x) : \|\nu\| = 1\}$, where $1 \leq s(f, x) \leq d$. This number $\lambda_i(f, x)$ is called the Lyapunov exponent associated with E^i . Lyapunov exponents describe the asymptotic evolution of tangent map: positive or negative exponents correspond to exponential growth or decay of the norm, respectively, whereas vanishing exponents mean lack of exponential behavior. None of these values depends on the choice of a Riemannian metric.

Throughout this paper, let $\lambda_1(f, x) \geq \lambda_2(f, x) \geq \dots \geq \lambda_d(f, x)$ be the Lyapunov exponents in nonincreasing order and each repeated with multiplicity $\dim E^i(x)$. We define **the j -th-integrated exponent** by $\int_M \lambda_j(f, x) d\omega(x)$, for any $1 \leq j \leq d$.

A Df -invariant splitting $TM = E^1 \oplus \dots \oplus E^k$ is called a *dominated splitting* if each E_i is a continuous Df -invariant subbundle of TM and if there is some integer $n > 0$ such that, for any $x \in M$, any $i < j$ and any non-zero vectors $u \in E^i(x)$ and $\nu \in E^j(x)$, one has

$$\frac{\|Df^n(u)\|}{\|u\|} < \frac{1}{2} \frac{\|Df^n(\nu)\|}{\|\nu\|}.$$

Let $\mathcal{PH}_\omega^1(M)$ denote the subset of $\text{Diff}_\omega^1(M)$ consisting of all the partially hyperbolic diffeomorphisms. In this paper, partially hyperbolic means the following. There is a nontrivial splitting of the tangent bundle, $TM = E^u \oplus E^c \oplus E^s$, that is invariant under the derivative map Df . Further, there is a Riemannian metric for which we can choose continuous positive functions $\nu, \tilde{\nu}, \gamma$ and $\tilde{\gamma}$ with

$$\nu, \tilde{\nu} < 1 \text{ and } \nu < \gamma < \tilde{\gamma}^{-1} < \tilde{\nu}^{-1}$$

such that, for any unit vector $v \in T_p M$,

$$\|Df v\| < \nu(p), \text{ if } v \in E^s(p), \quad (1.2)$$

$$\gamma(p) < \|Df v\| < \tilde{\gamma}(p)^{-1}, \text{ if } v \in E^c(p), \quad (1.3)$$

$$\tilde{\nu}(p)^{-1} < \|Df v\|, \text{ if } v \in E^u(p). \quad (1.4)$$

Theorem 1.1. *Let $f \in \mathcal{PH}_\omega^1(M)$. Then for any neighborhood $\mathcal{U} \subseteq \mathcal{PH}_\omega^1(M)$ of f , there is a diffeomorphism $g \in \mathcal{U}$, such that every integrated Lyapunov exponent is different from zero, i.e.,*

$$\int_M \lambda_j(g, x) d\omega(x) \neq 0, \quad \forall 1 \leq j \leq d.$$

Much work on perturbation of Lyapunov exponents concerns two basic topics. One is when we can remove zero Lyapunov exponents as we discussed above. The other is when we can distinguish all the Lyapunov exponents. That is, the spectrum is simple. About the latter topic, all work are concentrated for cocycles. In the case of independent and identically distributed random matrices,

Arnold and Cong[4] showed that the cocycles with simple Lyapunov spectrum form a residual set. In the space of all linear cocycles equipped with the uniform topology, Knill[25] (for the 2-dimensional case) and Arnold and Cong[5] (for the general d -dimensional case) proved that they form a dense set. For $SL(d, \mathbb{R})$ -cocycles, Bonatti and Viana[17] also showed the density property under some conditions. It is natural to ask whether the diffeomorphisms with simple spectrum form a dense set in $\text{Diff}_\omega^1(M)$. In Section 3, we construct an example related to this problem and show the local genericity of diffeomorphisms with non-simple spectrum. Using the techniques developed in [18] and [26], we prove a dichotomy between simple-spectrum property and the existence of complex eigenvalues.

Theorem 1.2. *There is a residual subset $\mathcal{R} \subset \mathcal{PH}_\omega^1(M)$ of diffeomorphisms f such that*

- *either the finest dominated splitting of E^c is of the form*

$$E^c(x) = \oplus_{i=1}^c E^{ci}(x) \text{ with } \dim E^{ci}(x) = 1, \text{ for all } i \in \{1, 2, \dots, c\}, \quad (1.5)$$

ω -a.e. $x \in M$, where $c = \dim E^c$. Hence, the multiplicity of every center Lyapunov exponent of f is 1;

- *or for any $\varepsilon > 0$, there is an ε -perturbation $g \in \mathcal{PH}_\omega^1(M)$ of f and a periodic point q of g such that $Dg_q^{p(q)}$ has a complex center eigenvalue, where $p(q)$ is the period of q .*

In the previous theorem, the first case is that each invariant subbundle in the finest dominated splitting is 1-dimensional. For the center submanifold, this property is much stronger than that of simple spectrum. Results in [34][28][27] indicates that, roughly speaking, dominated splitting and homoclinic tangency are mutually exclusive concepts. We prove two results which suggest that diffeomorphisms far from homoclinic tangencies and in the interior of the complement load bearing entirely different properties. Moreover, applying the novel approach developed by F.R.Hertz, M.A.R.Hertz, A.Tahzibi, and R.Ures[23, 24] for Pugh-Shub's stable ergodicity conjecture, we point out the genericity of the ergodic diffeomorphisms in the conservative ones which are far from tangencies.

We say a diffeomorphism f has C^1 homoclinic tangency if $f \in \text{Diff}^1(M)$ has hyperbolic periodic point p at which the stable and unstable invariant manifolds $W^s(p)$ and $W^u(p)$ intersect non-transversely. Denote by HT the set containing all the diffeomorphisms with homoclinic tangencies. We call a diffeomorphism f is far away from tangency if $f \in \text{Diff}^1(M) \setminus \overline{HT}$.

In the context of the following two results, we suppose $d = \dim M \geq 3$.

Theorem 1.3. *There is a C^1 residual subset \mathcal{R} of the volume preserving diffeomorphisms far from tangencies, such that any $f \in \mathcal{R}$ admits a finest dominated splitting $E^s \oplus E^{c1} \oplus \dots \oplus E^{cc} \oplus E^u$, where $\dim(E^{ci}) = 1$ and $c = \dim E^c$. Moreover, f is ergodic.*

Corollary 1.4. *There is a C^1 residual subset \mathcal{R} of the volume preserving diffeomorphisms far from tangencies, such that each center Lyapunov exponent of $f \in \mathcal{R}$ has multiplicity 1.*

Recall a partially hyperbolic diffeomorphism is center bunched if the functions ν , $\tilde{\nu}$, γ and $\tilde{\gamma}$ in (1.2)-(1.4) can be chosen so that:

$$\nu < \gamma\tilde{\gamma} \quad \text{and} \quad \tilde{\nu} < \gamma\tilde{\gamma}.$$

Let $\mathcal{CPH}_\omega^1(M)$ denote the set of all center bunching, conservative, partially hyperbolic diffeomorphisms. The statement of the third result about spectrum is the following:

Theorem 1.5. *There is a C^1 residual subset of diffeomorphism \mathcal{R} in $\mathcal{CPH}_\omega^1(M) \cap \text{int}(\overline{HT})$, such that for any $f \in \mathcal{R}$, it is ergodic, and it has two center exponents equal.*

2 Proof of Theorem 1.1

Given a vector space V and a positive integer p , let $\wedge^p(V)$ be the p -th exterior power of V . This is a vector space of dimension C_d^p , whose elements are called p -vectors. It is generated by the p -vectors of the form $\nu_1 \wedge \cdots \wedge \nu_p$ with $\nu_j \in V$, called the *decomposable p -vectors*. A linear map $L : V \rightarrow W$ induces a linear map $\wedge^p(L) : \wedge^p(V) \rightarrow \wedge^p(W)$ such that

$$\wedge^p(L)(\nu_1 \wedge \cdots \wedge \nu_p) = L(\nu_1) \wedge \cdots \wedge L(\nu_p).$$

If V has an inner product, then we endow $\wedge^p(V)$ with the inner product such that $\|\nu_1 \wedge \cdots \wedge \nu_p\|$ equals the p -dimensional volume of the parallelepiped spanned by ν_1, \dots, ν_p .

More generally, there is a vector bundle $\wedge^p(\mathcal{E})$, with fibers $\wedge^p(\mathcal{E}_x)$, associated to \mathcal{E} , and there is a vector bundle automorphism $\wedge^p(F)$, associated to F . If the vector bundle \mathcal{E} is endowed with a continuous inner product, then $\wedge^p(\mathcal{E})$ also is. The Oseledec data of $\wedge^p(F)$ can be obtained from that of F , as shown by the proposition below.

Proposition 2.1 (Theorem 5.3.1 in [3]). *The Lyapunov exponents (with multiplicity) $\lambda_i^{\wedge^p}(x)$, $1 \leq i \leq C_d^p$, of the automorphism $\wedge^p(F)$ at a point x are the numbers*

$$\lambda_{i_1}(x) + \cdots + \lambda_{i_p}(x), \quad \text{where } 1 \leq i_1 < \cdots < i_p \leq d.$$

Let $\{e_1(x), \dots, e_d(x)\}$ be a basis of \mathcal{E}_x such that

$$e_i(x) \in E_x^\ell \quad \text{for } \dim E_x^1 + \cdots + \dim E_x^{\ell-1} < i \leq \dim E_x^1 + \cdots + \dim E_x^\ell.$$

Then the Oseledec space E_x^{j, \wedge^p} of $\wedge^p(F)$ corresponding to the Lyapunov exponent $\hat{\lambda}_j(x)$ is the sub-space of $\wedge^p(\mathcal{E}_x)$ generated by the p -th vectors

$$e_{i_1} \wedge \cdots \wedge e_{i_p}, \quad \text{with } 1 \leq i_1 < \cdots < i_p \leq d \quad \text{and } \lambda_{i_1} + \cdots + \lambda_{i_p} = \hat{\lambda}_j(x).$$

A dominated splitting $E^1 \oplus \cdots \oplus E^n$ is called the finest dominated splitting if there is no dominated splitting defined over all M in each invariant bundle E^i , $1 \leq i \leq n$. The continuation of the finest dominated splitting may not be the finest dominated splitting of the perturbation diffeomorphism. However, we can perturb any partially hyperbolic diffeomorphism to obtain a diffeomorphism with robust finest dominated splitting.

Lemma 2.2. *The diffeomorphisms with robust finest dominated splitting are C^1 dense among the C^r , $r \geq 1$, partially hyperbolic diffeomorphisms of M .*

proof For any neighborhood \mathcal{U} of f in $\mathcal{PH}_\omega^1(M)$, we denote $\mathcal{U}_f \subset \mathcal{U}$ be the neighborhood of f in $\mathcal{PH}_\omega^1(M)$ such that the continuation of the dominated splitting of f is a dominated splitting of the perturbation diffeomorphism. Assume that there is a small perturbation $h_1 \in \mathcal{U}_f$ such that the continuation of the finest dominated splitting of f is not the finest one of h_1 . Then the finest dominated splitting of h_1 must be a refinement of the continuation. Consider the subset $\mathcal{U}_{h_1} \cap \mathcal{U}_f$. If there exists another perturbation h_2 in $\mathcal{U}_{h_1} \cap \mathcal{U}_f$ whose finest dominated splitting is a refinement of the continuation of the finest dominated splitting of h_1 , we shift our attention to a smaller

subset $\mathcal{U}_{h_2} \cap \mathcal{U}_{h_1} \cap \mathcal{U}_f$. The process will stop at some perturbation h_n since the number of invariant bundles in the finest dominated splitting of h_i strictly increases (as i increases) and this number can not exceed d , the dimension of M . Let $g = h_n$. Then g must has robust finest dominated splitting.

□

The main techniques in the proof of Theorem 1.1 are three results as we cite in the following:

A partially hyperbolic diffeomorphism f is accessible if, for every pair of points $p, q \in M$, there is a C^1 path from p to q whose tangent vector always lies in $E^u \cup E^s$ and vanishes at most finitely many times. We say f is stably accessible if every g sufficiently C^1 -close to f is accessible.

Lemma 2.3 (Main Theorem in [21]). *For any $r \geq 1$, stable accessibility is C^1 dense among the C^r , partially hyperbolic diffeomorphisms of M , volume preserving or not.*

Lemma 2.4 (Theorem 2 in [8]). *Let M be a compact manifold and ω be a smooth volume form. Let f be an ω -preserving C^1 -diffeomorphism of M , admitting a dominated splitting $TM = E^1 \oplus \dots \oplus E^k$, $k > 1$.*

Then there are ω -preserving diffeomorphisms g , arbitrarily C^1 -close to f , for which the integral $\int_M \log |Det Dg|_{E^i}(x)| d\omega(x)$ is different from 0 for each $i \in \{1, \dots, k\}$.

Let $\mathcal{D}_p(f, m)$ be the set of points x such that there is an m -dominated splitting of index p (i.e., $p = \dim E^s$) along the orbit of x . Then $\mathcal{D}_p(f, m)$ is a closed set. Define

$$\Gamma_p(f, m) = M \setminus \mathcal{D}_p(f, m)$$

and

$$\Gamma_p(f, \infty) = \bigcap_{m \in \mathbb{N}} \Gamma_p(f, m).$$

We recall from Proposition 2.1 that for any diffeomorphism $f \in \text{Diff}_\omega^1(M)$ and $p \in \{1, \dots, d-1\}$, the integrated Lyapunov exponent of $\wedge^p(Df)$ coincides with the sum of the largest p integrated Lyapunov exponents, that is,

$$\int_M \wedge^p(Df) d\omega = \int_M \wedge_p(f, x) d\omega(x),$$

where $\wedge_p(f, x)$ denotes the sum, i.e., $\wedge_p(f, x) = \lambda_1(f, x) + \dots + \lambda_p(f, x)$.

Lemma 2.5 (Proposition 4.17 in [11]). *Let $f \in \text{Diff}_\omega^1(M)$ and $p \in \{1, \dots, d-1\}$, and given any $\varepsilon_0 > 0$ and $\delta > 0$, there exist a neighborhood $\mathcal{U}(f, \varepsilon_0) \subset \text{Diff}_\omega^1(M)$ of f with radius ε_0 and a diffeomorphism $g \in \mathcal{U}(f, \varepsilon_0)$ such that*

$$\int_M \wedge_p(g, x) d\omega(x) < \int_M \wedge_p(f, x) d\omega(x) - J_p(f) + \delta.$$

where $J_p(f) = \int_{\Gamma_p(f, \infty)} \frac{\lambda_p(f, x) - \lambda_{p+1}(f, x)}{2} d\omega(x)$.

Proof of Theorem 1.1

By Lemma 2.3, we can perturb f to a diffeomorphism $f_1 \in \mathcal{PH}_\omega^1(M)$ with stable accessibility. Take a neighborhood $\mathcal{U}_1 \subset \mathcal{PH}_\omega^1(M)$ of f_1 such that each diffeomorphism in \mathcal{U}_1 is accessible. Applying Lemma 2.2, we get another small perturbation $f_2 \in \mathcal{U}_1$ which is accessible and has the robust finest dominated splitting:

$$TM = E^s \oplus E^{c1} \oplus \dots \oplus E^{ck} \oplus E^u, \quad 1 \leq k \leq d = \dim M \quad (2.1)$$

(Since any integrated Lyapunov exponent related to E^s and E^u is robustly away from zero, we need not care about the decomposition of E^s and E^u). Therefore, there is a neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of f_2 such that any $g \in \mathcal{U}_2$, g is accessible and has the same finest dominated splitting as f_2 . By perturbing f_2 if necessary, Lemma2.4 ensures that there exists a neighborhood $\mathcal{U}_3 \subset \mathcal{U}_2$ with property:

$$\int_M \log |Det Dg|_{E^{ci}}(x) d\omega(x) \neq 0, \quad 1 \leq i \leq k, \quad \forall g \in \mathcal{U}_3. \quad (2.2)$$

That is, the sum of the integrated Lyapunov exponents

$$\sum_{j=s+c_{i-1}+1}^{s+c_i} \lambda_j(g, x) d\omega(x) \neq 0, \quad 1 \leq i \leq k$$

where we denote $s = \dim E^s$ and $u = \dim E^u$ and $c_i = \dim E^{ci}$ and $c_0 = 0$ for simplicity.

Notice that

$$\int_M \Lambda_p(g, x) d\omega(x) = \inf_n \frac{1}{n} \int_M \log \|\Lambda^p(Dg^n)\| d\omega(x).$$

The functions $g \in \text{Diff}_\omega^1(M) \mapsto \int_M \Lambda_p(g, x) d\omega(x)$ are upper semi-continuous for any $1 \leq p \leq d$. Hence the continuity points of the map

$$g \in \text{Diff}_\omega^1(M) \mapsto \left(\int_M \Lambda_1(g, x) d\omega(x), \dots, \int_M \Lambda_d(g, x) d\omega(x) \right) \quad (2.3)$$

form a residual subset. We choose a continuity point h of the above map in \mathcal{U}_3 . Now we verify that h meets the requirements of our theorem.

Since $h \in \mathcal{U}_3 \subset \mathcal{U}_1$, h is accessible. This implies that for ω -almost every $x \in M$, the orbit of x is dense in M . In fact, Burns-Dolgopyat-Pesin[18] pointed out in the proof of Theorem 2 that the essential accessibility property indicates that almost every point has a dense orbit. Note that the essential accessibility property is a weaker property than accessibility. Precisely, recall that accessibility is an equivalence relation. If a diffeomorphism is accessible then the partition into accessibility classes is trivial. A diffeomorphism is said to be essentially accessible if the partition into accessibility classes is ergodic (i.e. a measurable union of equivalence classes must have zero or full measure). From the definition, one can deduce that an accessible diffeomorphism is also essentially accessible. Hence h is essentially accessible, since h is accessible. And then almost every point has a dense orbit.

Then for any $s+1 \leq p \leq d-u$, we have either $D_p(h, m) = M \bmod 0$ for some integer $m > 0$, or $\Gamma_p(h, \infty) = M \bmod 0$. In fact, if one has that $\omega(D_p(h, m)) > 0$ for some integer $m > 0$, by the accessibility, for ω -almost every point in $D_p(h, m)$, we can spread the dominated splitting along its orbit to the closure, that is, the whole manifold M . Since h has the same finest dominated splitting as f_2 , we have that

$$\Gamma_p(h, \infty) = M \bmod 0, \quad s + c_{i-1} + 1 \leq p \leq s + c_i, \quad \forall 1 \leq i \leq k$$

and

$$D_p(h, m) = M \bmod 0, \quad p = c_i, \quad \forall 1 \leq i \leq k.$$

Note that h is a continuity point of map (2.3). Combining with Lemma2.5, one can obtain that $J_p(h) = 0$, i.e.,

$$\lambda_p(h, x) = \lambda_{p+1}(h, x) \quad \forall s + c_{i-1} + 1 \leq p \leq s + c_i, \quad \forall 1 \leq i \leq k.$$

Therefore, one can deduce from (1.3) that

$$\int_M \lambda_p(h, x) d\omega(x) = \frac{1}{c_i} \int_M \log |Det Dh|_{E^{ci}}(x) d\omega(x) \neq 0, \quad \forall s + c_{i-1} + 1 \leq p \leq s + c_i, \quad \forall 1 \leq i \leq k.$$

□

Remark When we have done this paper, we find [23] which is considering the same problem in Theorem 1.1 of the case $\dim E^c = 2$.

3 The example and Proofs of Theorems 1.2-1.5

In this section we will first present an example of a linear Anosov diffeomorphism A on \mathbb{T}^3 which has a couple of complex eigenvalues. Then we show that for any small neighborhood of A in $\text{Diff}_\omega^1(\mathbb{T}^3)$, there exists a residual subset consisting of diffeomorphisms with non-simple spectrum. Then we will give the proof of Theorems 1.2, 1.3 and 1.5.

3.1 The example

Example: Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} : \mathbb{T}^3 \rightarrow \mathbb{T}^3,$$

Then A is stable ergodic and

$$\det(A - \lambda Id) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & 1 - \lambda \end{vmatrix} = \lambda^2(1 - \lambda) + 1 = -\lambda^3 + \lambda^2 + 1 = f(\lambda).$$

Since $f(1) = 1 > 0$ and $f(2) = -3 < 0$, by the continuity of f , there is $c \in (1, 2)$, such that $f(c) = 0$.

On the other hand, one consider the converse matrix

$$A^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and the determinant } \det(A^{-1} - \lambda Id) = -\lambda^3 - \lambda + 1 = g(\lambda).$$

We have that $g(0) = 1 > 0$ and $g(1) = -1 < 0$. This implies that $\exists d \in (0, 1)$, such that $g(d) = 0$. Note that $g'(\lambda) = -3\lambda^2 - 1 < 0$. Therefore, the point d is the unique real root of $g(\lambda)$ and hence the other two eigenvalues are not real. Moreover, observing that the coefficients of the first and last items are both 1, we can deduce that d must be an irrational number.

Note that when a number λ is an eigenvalue of A , its reciprocal $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and conversely, it also holds. Thus we have the following conclusion about A .

Conclusion I. $\exists p \in \mathbb{T}^3$ (hyperbolic fixed point) has complex eigenvalues $\lambda_1, \overline{\lambda_1}$ and a real (irrational) eigenvalue λ_2 satisfying

$$|\lambda_1| < 1 < |\lambda_2|.$$

This conclusion implies that A has non-simple spectrum. Moreover, applying the following lemma, we will show that for any small neighborhood of A in $\text{Diff}_\omega^1(\mathbb{T}^3)$, there is a residual subset composed of diffeomorphisms with non-simple spectrum.

Lemma 3.1 (Theorem 1 in [11]). *There exists a residual set $\mathcal{R} \subset \text{Diff}_\omega^1(M)$ such that, for each $f \in \mathcal{R}$ and ω -almost every $x \in M$, the Oseledec splitting of f is either trivial or dominated at x .*

We choose a neighborhood $\mathcal{U} \subset \text{Diff}_\omega^1(\mathbb{T}^3)$ of A , such that $\forall g \in \mathcal{U}$, g is ergodic and has a fixed point (i.e., the continuation of p) with complex eigenvalues. Using Lemma 3.1, there is a residual set $\mathcal{R} \subset \mathcal{U}$, such that $\forall g \in \mathcal{R}$, the Oseledec decomposition of g is dominated or trivial. Therefore, $\forall g \in \mathcal{R}$, if g has simple spectrum, its dominated splitting must be the form

$$E_1(x) \oplus E_2(x) \oplus E_3(x)$$

for every generic point $x \in O(\omega)$, the Oseledec basin of ω , since g is ergodic. Note that ω is a volume form and the Oseledec basin has ω -full measure, the Oseledec basin is dense in M . Thus the dominated splitting $E_1(x) \oplus E_2(x) \oplus E_3(x)$ of g can be extended to the whole manifold M by the continuity of dominated splitting. This contradicts with the property that g has complex eigenvalues. Thus we have the second conclusion about the local genericity of diffeomorphisms with non-simple spectrum.

Conclusion II. For any small neighborhood $\mathcal{V} \subset \text{Diff}_\omega^1(\mathbb{T}^3)$ of A , there exists a residual subset $\mathcal{R} \subset \mathcal{V}$ consisting of diffeomorphisms with non-simple spectrum.

3.2 The proof of Theorem 1.2

Bonatti and Crovisier[12] proved that, generically, a volume-preserving diffeomorphism is transitive in a compact, connected manifold with a volume-preserving diffeomorphism.

Lemma 3.2 (Thm1.3 in [12]). *Suppose M is connected. Then there is a residual subset G_ω in $\text{Diff}_\omega^1(M)$ such that any $f \in G_\omega$ is transitive. Moreover, M is the unique homoclinic class.*

To prove Theorem 1.2, we need introduce some preliminary notions and lemmas first.

Definition 3.3. *We shall call any 4-uple $\mathcal{A} = (\Sigma, f, \mathcal{E}, A)$ to be a linear cocycle of dimension d if:*

- Σ is a set and $f : \Sigma \rightarrow \Sigma$ is a one-to-one map;
- $\pi : \mathcal{E} \rightarrow \Sigma$ is a linear bundle of dimension d over Σ , whose fibers are endowed with an Euclidean metric $\|\cdot\|$. The fiber over the point $x \in \Sigma$ will be denoted by \mathcal{E}_x ;
- $A : x \in \Sigma \mapsto A_x \in GL(\mathcal{E}_x, \mathcal{E}_{f(x)})$ is a map.

We shall say that a linear cocycle A is bounded if there exists a constant $K > 0$ such that, for any $x \in \Sigma$, we have $\|A_x\| < K$ and $\|A^{-1}x\| < K$. We call K a bound of \mathcal{A} .

Remark Here the base space Σ can be taken arbitrarily, even not a topology space.

Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system, an invariant subbundle is a collection of linear subspaces $F(x) \subset \mathcal{E}_x$ whose dimensions do not depend on x and such that $A(F(x)) = F(f(x))$. An A -invariant splitting $F \oplus G$ is given by two invariant subbundles such that $\mathcal{E}_x = F(x) \oplus G(x)$ at each $x \in \Sigma$.

Definition 3.4. Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system and $\mathcal{E} = F \oplus G$ an A -invariant splitting. We say that $F \oplus G$ is a dominated splitting if there exists $n \in \mathbb{N}$ such that

$$\|A^{(n)}(x)|_F\| \|A^{(-n)}(f^n(x))|_G\| < 1/2$$

for every $x \in \Sigma$. We write $F \prec G$.

If we want to emphasize the role of n then we say that $F \oplus G$ is an n -dominated splitting and write $F \prec_n G$. Finally, the dimension of the dominated splitting is the dimension of the subbundle F .

Definition 3.5. Let A be a linear map on a d -dimensional linear space. The complex eigenvalues $(\lambda, \bar{\lambda})$ of A is called of rank $(i, i+1)$, where $1 \leq i \leq d-1$, if the moduli of all its other eigenvalues are different from $|\lambda|$ and the number of the eigenvalues which are less than $|\lambda|$ coincide with $i-1$.

The following is Theorem 3.6 in [26] with more details.

Lemma 3.6 (Theorem 3.6 in [26]). Given $K > 0$ and $\varepsilon > 0$ there is $\ell \in \mathbb{N}$ such that for any linear periodic system $\mathcal{A} = (\Sigma, f, \mathcal{E}, A)$ bounded by K , one has that it admits a finest dominated splitting $E = E_1 \oplus_{\prec_\ell} E_2 \oplus_{\prec_\ell} \cdots \oplus_{\prec_\ell} E_k$ if and only if we can not get a complex eigenvalue of the following ranks

$$(\tau_1, \tau_1 + 1), (\tau_2, \tau_2 + 1), \dots, (\tau_{k-1}, \tau_{k-1} + 1)$$

by any ε -perturbation of \mathcal{A} , where $\tau_i = \sum_{j=1}^i \dim E_j, i = 1, 2, \dots, k$.

Proof of Theorem 1.2: Let G_ω be the residual subset of $\text{Diff}_\omega^1(M)$ determined by Lemma 3.2. Define

$$\mathcal{R} = G_\omega \cap \mathcal{PH}_\omega^1(M).$$

Then \mathcal{R} is a residual subset $\mathcal{PH}_\omega^1(M)$ of diffeomorphisms f such that M is the unique homoclinic class.

For any diffeomorphism $f \in \mathcal{R}$, we take $K = \max_{x \in M} \{\|Df|_{E^c}\|, \|Df^{-1}|_{E^c}\|\}$. Then we obtain a positive integer ℓ by Lemma 3.6. If E^c admits an ℓ -finest dominated splitting as the form (1.5), the proof is done. Otherwise, there is an integer $s < i < s + c$, such that $E^c(x)$ has no dominated decomposition $E^c(x) = E(x) \oplus_{\prec} F(x)$ with $\dim E = i - s$ for x in some positive-measure subset of M . Then there are periodic points such that the cocycles defined over these points do not have dominated decomposition $E^c = E \oplus_{\prec_n} F$ with $\dim E = i - s$ and uniform time n , for any integer $n \in \mathbb{N}$. Let

$$\Sigma = M, \mathcal{E} = E^c, A = Df|_{E^c}.$$

Then $\mathcal{A} = (\Sigma, f, \mathcal{E}, A)$ is a linear cocycle bounded by K over an infinite periodic system having transitions but without dominated splitting $E^c = E \oplus_{\prec} F$ where $\dim E = i - s$. Now using Lemma 3.6, for any $\varepsilon > 0$, we obtain a ε -perturbation B of A and a periodic point in Σ , such that $M_{q,B} = B(f^{p(q)-1}q) \circ \cdots \circ B(q)$ has a complex eigenvalue of rank $(i, i+1)$. Applying Franks' Lemma, we get a C^1 -perturbation $g \in \mathcal{PH}_\omega^1(M)$ of f , which coincides with f out of an arbitrarily small neighborhood of $\text{Orb}(q)$, equals to f in $\text{Orb}(q)$, and whose derivation satisfying

$$Dg|_{E^c(g, f^i q)} = B_{f^i q}, i = 0, 1, \dots, p(q) - 1.$$

Hence $Dg_q^{p(q)}$ has a complex eigenvalue. □

3.3 The proof of Theorem 1.3

In this subsection, we focus on conservative diffeomorphisms far from homoclinic tangencies and discuss their two generic properties: one is the property of admitting a dominated splitting as (1.5) (Lemma 3.7) and the other is ergodicity (Lemma 3.10).

Lemma 3.7. *For C^1 generic $f \in \text{Diff}_\omega^1(M)$, ($d = \dim M \geq 3$), it is C^1 far away from tangencies if and only if there exists a dominated splitting $E^s \oplus E^c \oplus E^u$ with two non-trivial extreme subbundles and the finest dominated splitting of the center bundle is of the form*

$$E^c(x) = \oplus_{i=1}^c E^{ci}(x) \text{ with } \dim E^{ci}(x) = 1, \text{ for all } i \in \{1, 2, \dots, c\}, \quad (1.6)$$

$\omega - a.e. x \in M$, where $c = \dim E^c$.

Abdenur-Bonatti-Crovisier-Diaz-Wen claim that, for C^1 -generic diffeomorphisms, the set of indices of the (hyperbolic) periodic points in a chain recurrence class (in fact, such classes are homoclinic ones) form an interval in \mathbb{N} . Applying the connecting lemma and Franks' lemma for conservative diffeomorphisms, we obtain the conservative version of this result (Theorem 1.1 in [1]) as follows:

Lemma 3.8 (the conservative version of Thm 1.1 in [1]). *There is a residual subset I of $\text{Diff}_\omega^1(M)$ of diffeomorphisms f such that, for every $f \in I$, any homoclinic class $H(p, f)$ containing hyperbolic saddles of indices α and β contains a dense subset of saddles of index τ for all $\tau \in [\alpha, \beta] \cap \mathbb{N}$.*

Recall that a point $x \in M$ is called (C^1) i -preperiodic of f , $0 \leq i \leq d$, if for any neighborhood \mathcal{U} of f in $\text{Diff}^1(M)$ and any neighborhood U of x in M , there exist $g \in \mathcal{U}$ and $y \in U$ such that y is a hyperbolic periodic point of g with index i . Now we begin to prove one of the main lemma in this subsection:

Proof of Lemma 3.7: Let

$$\mathcal{R} = G_\omega \cap I,$$

where G_ω and I are determined by Lemma 3.2 and 3.8, respectively. For $f \in \mathcal{R}$, let i_0 (resp i_1) be the minimal (resp. maximal) preperiodic index of f . Then $i_0 \geq 0$ and $i_1 \leq d$. By the definition of preperiodic points, we can assume that f itself contains index i_0, i_1 periodic points. By Lemma 3.8, f contains i -index periodic points for all $i \in [i_0, i_1] \cap \mathbb{N}$. Since M is the unique homoclinic class by Lemma 3.2 and f is far from tangencies, now we obtain a dominated splitting with center bundles all 1 dimensional by the equivalent conditions for the existence of dominated splitting (for more details, see [28, 34]).

Since f is conservative, one must have that $i_0 \geq 1$ and $i_1 \leq d - 1$. Otherwise, if $i_0 = 0$, the minimal of periodic index can be 0 or 1 (There may exist a weak stable subbundle with dimension 1). In the first case, all the Lyapunov exponents of f are non-negative. Hence the sum of the Lyapunov exponents are nonzero. This contradicts with the conservative condition. In the second case, the negative exponent have multiplicity 1 and is close to 0, but the positive exponents are uniformly far from 0 by the domination. This again deduces the same contradiction as the first case. The proof of $i_1 \leq d - 1$ is analogous.

The rest thing is to prove the two extreme bundles are uniformly hyperbolic. This can be obtained by showing that it is forbidden to decrease the index of a periodic point with index i_0 or increase that of a periodic point with index i_1 by perturbation. The proof essentially from Mañé's Ergodic Closing Lemma [29], in conservative version (see also [36] or Theorem B and Section 4 in [15]). This concludes the proof of necessity.

The sufficiency is ensured by the uniform hyperbolicity of the extreme subbundles and the domination of the splitting (1.6) of the center bundle. This ends the whole proof. \square

Remark: Dawei Yang told us the analogous consequence for dissipative dynamics of Lemma 3.7 may be deduced by his recent joint work with Crovisier and Sambrino. And he suggested us to omit the original hypothesis of partial hyperbolicity.

Recall that the integers s , c and u denote the dimensions of f -invariant subbundles E^s , E^c and E^u in the dominated splitting $TM = E^s \oplus E^c \oplus E^u$. To emphasis their dependence on f , we write them as $s(f)$, $c(f)$ and $u(f)$. Since (G1) – (G4) in the following proposition are open properties, we combine the preceding lemma with Lemma 3.8, 2.3 and Theorem 1.1 and obtain that:

Proposition 3.9. *There is a C^1 open and dense subset \mathcal{O} of the volume preserving diffeomorphisms far from tangencies, such that any $f \in \mathcal{O}$ satisfies the following properties.*

- (G1) *f is partially hyperbolic, it admits a partially hyperbolic splitting admits a finest dominated splitting $E^s \oplus E^{c1} \oplus \dots \oplus E^{c,c(f)} \oplus E^u$, where $\dim(E^{ci}) = 1$.*
- (G2) *f has periodic points with index $\dim(E^s), \dim(E^s) + 1, \dots, \dim(E^s) + c(f)$.*
- (G3) *f is accessible.*
- (G4) *There is $0 \leq j(f) \leq c(f)$, such that for any integer $1 \leq j \leq j(f)$, one has*

$$\int \log \|Df|_{E^{c,j}(x)}\| d\omega(x) < 0,$$

and for any integer $j(f) + 1 \leq j \leq c(f)$, we has

$$\int \log \|Df|_{E^{c,j}(x)}\| d\omega(x) > 0.$$

When $j(f) = 0$ (resp. $j(f) = c(f)$), we take $E^{c,j(f)}(x)$ (resp. $E^{c,j(f)+1}(x)$) to be vanished.

Recently, F.R.Hertz, M.A.R.Hertz, A.Tahzibi, and R.Ures[23, 24] developed a new criteria of ergodicity and nonuniform hyperbolicity which provided fresh ideas to the Pugh-Shub stable ergodicity conjecture. Applying Proposition 3.9 and their criteria, we illustrate the density of ergodicity as the follow which implies the ergodic diffeomorphisms form a residual subset, since it is a G_δ set.

Lemma 3.10. *There is a C^1 dense subset of diffeomorphisms \mathcal{E} in \mathcal{O} , such that any diffeomorphism in \mathcal{E} is ergodic.*

Before give the proof of Lemma 3.10, we need give some definitions and notations.

• **Existence of stable manifolds**

For each point x in a compact Riemannian manifold M , the Pesin stable manifold of x is

$$W^s(x) = \{y \in M : \limsup_{n \rightarrow \infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0\}$$

and the Pesin unstable manifold of x , $W^u(x)$, is the Pesin stable manifold for f^{-1} . In the context, we use \mathcal{W}^s (resp. \mathcal{W}^u) to denote the strong stable (resp. unstable) foliation.

For any $i \in \mathbb{N}$, let $I_1^i = (-1, 1)^i$ and $I_\varepsilon^i = (-\varepsilon, \varepsilon)^i$ and denote by $Emb^1(I_1^i, M)$ the set of C^1 -embeddings of I_1^i on M . Suppose a compact invariant set Λ admits a dominated splitting $E \oplus F$. The following lemmas are Lemma 3.0.4 and Corollary 3.3 in [31] for C^1 case with high dimension which come from the existence of the dominated splitting on Λ .

Lemma 3.11 (Lemma 3.0.4 in [31]). *There exist two continuous functions*

$$\Phi^{cs} : \Lambda \longrightarrow Emb^1(I_1^{dim E}, M) \quad \text{and} \quad \Phi^{cu} : \Lambda \longrightarrow Emb^1(I_1^{dim F}, M)$$

such that, with $W_\varepsilon^{cs}(x) = \Phi^{cs}(x)I_\varepsilon^{dim E}$ and $W_\varepsilon^{cu}(x) = \Phi^{cu}(x)I_\varepsilon^{dim F}$, the following properties hold:

- (a) $T_x W_\varepsilon^{cs}(x) = E(x)$ and $T_x W_\varepsilon^{cu}(x) = F(x)$;
- (b) *for any $0 < \varepsilon_1 < 1$, there exists ε_2 such that $f(W_{\varepsilon_2}^{cs}(x)) \subset W_{\varepsilon_1}^{cs}(f(x))$ and $f^{-1}(W_{\varepsilon_2}^{cu}(x)) \subset W_{\varepsilon_1}^{cu}(f^{-1}(x))$;*

Corollary 3.12 (Corollary 3.3 in [31]). *For $0 < \lambda < 1$, there exists $\varepsilon > 0$ such that, for any $x \in \Lambda$ satisfying $\prod_{j=0}^{n-1} \|Df|_{E(f^j(x))}\| \leq \lambda^n$ for all $n > 0$, one has $\text{diam}(f^n(W_\varepsilon^{cs}(x))) \longrightarrow 0$.*

As a corollary of Lemma 3.11 and Corollary 3.12, we have the following proposition:

Proposition 3.13. *Let f be a volume preserving diffeomorphism. Suppose f admits a dominated splitting $E^{cs} \oplus E_1^c \oplus E^{cu}$ with $\dim(E_1^c) = 1$ and the set $\Lambda^s = \{x; \lambda^c(x) < 0\}$ has positive volume. Then Lebesgue almost every point $x \in \Lambda^s$ has a local stable manifold $W_{loc}^s(x)$ with dimension $E^{cs} \oplus E_1^c$. Moreover, $W_{loc}^s(x)$ is tangent to the bundle $E^{cs} \oplus E_1^c$ (this means, then the tangent space of $W_{loc}^s(x)$ is contained in a small cone around $E^{cs} \oplus E_1^c$).*

Remark 3.14. *Suppose Λ admits a partially hyperbolic splitting $E^s \oplus E^{cs} \oplus E^c \oplus E^u$ and there is a local strong stable leaf $\mathcal{W}_{loc}^s(x)$ tangent to $E^s(x)$ at some point x . If the center Lyapunov exponent at x is negative (resp. positive), then $\mathcal{W}_{loc}^s(x) \subset W_\varepsilon^s(x)$ (resp. $\mathcal{W}_{loc}^u(x) \subset W_\varepsilon^u(x)$).*

• **Blenders**

Topologically blenders are Cantor sets with distinctive geometric feature which is introduced by C. Bonatti and L.J.Diaz in [13] to give a slightly different mechanism for constructing non-Axiom A diffeomorphisms and robust heterodimensional cycles. Later, the notion of blender was motivated by L.J.Diaz in [20] and developed by F.Herz-M.A.Hertz-A.Tahzibi-R.Ures in [23] to produce a local source of stable ergodicity.

Definition 3.15. Let q, p be hyperbolic periodic points of a diffeomorphism f with stable index i and $i + 1$ respectively.

We say that f has a cs -blender of index i associated to (q, p) if there is a C^1 neighborhood \mathcal{U} of f such that, for every $g \in \mathcal{U}$, one has $W^s(p_g)$ is contained in the closure of $W^s(q_g)$, where q_g, p_g are the analytic continuation of q and p for g .

Define a cu -blender in an analogous way, by concerning f^{-1} .

There are several definitions of blender given in [13], [16], [20] and [24]. Our definition comes from the Definition and Lemma 1.10 in page 369 of [13]. And the proposition below which partially comes from the main result of [24] holds for our definition.

Proposition 3.16. Let $f : M \rightarrow M$ be a diffeomorphism admitting a dominated splitting $E^{cs} \oplus E_1^c \oplus E^{cu}$ with $\dim(E^{cs}) = i > 0$ and $\dim(E_1^c) = 1$. Suppose f has a cs -blender of index i associated to (q, p) . Then there is a C^1 neighborhood \mathcal{U} of f , such that for any $g \in \mathcal{U}$, and for every $i + 1$ -dimension disk D which is tangent to a cone around the bundle $E^{cu} \oplus E_1^c$, we have that

$$D \pitchfork W^s(p_g) \neq \emptyset \quad \text{implies} \quad D \pitchfork W^s(q_g) \neq \emptyset,$$

where q_g, p_g are the analytic continuation of q and p for g .

Proof. We only need to notice that $W^s(q)$ is tangent to the bundle E^{cs} . From the definition of cs -blender, one can conclude the proof. \square

Definition 3.17. A diffeomorphism f is called to have a chain of cs -blenders of index (i_0, \dots, i_1) if

- (a) for any $i_0 \leq i \leq i_1$, f has a cs -blender of index i associated to periodic points (q_i, p_i) ,
- (b) for any $i_0 + 1 \leq i \leq i_1$, q_i is homoclinically related to p_{i-1} .

It is remarkable that the distinctive blender property is a robust property. And homoclinical relation is also an open property. Hence, if a diffeomorphism f has a chain of blenders of index (i_0, \dots, i_1) , there is a neighborhood \mathcal{U} of f , such that any $g \in \mathcal{U}$ has a chain of blenders of index (i_0, \dots, i_1) as well.

Connecting lemma was proved by Hayashi [22] at first, and then was extended to the conservative setting by Wen and Xia [37, 35] (see also M.-C. Arnaud[2]). Later, from the proof of Hayashi's Connecting Lemma, Bonatti and Crovisier[12] extract a slightly stronger statement in the next lemma which permits to perform dynamically perturbations and create intersections between stable and unstable bundles.

Lemma 3.18 (Theorem 2.1(Connecting lemma) in [12], Theorem 3.10 in [24]). Let p, q be hyperbolic periodic points of a C^r transitive diffeomorphism preserving a smooth measure m . Then, there exists a C^1 -perturbation $g \in C^r$ preserving m such that $W^s(p) \cap W^u(q) \neq \emptyset$.

Lemma 3.2 ensures the transitivity of the diffeomorphisms we are considering. Combining with the previous lemma, we can prove the following proposition.

Proposition 3.19. There is a C^1 open and dense subset $\mathcal{O}_1 \subset \mathcal{O}$ of diffeomorphisms such that for any $f \in \mathcal{O}_1$, it admits a dominated splitting $E^s \oplus E^{c1} \oplus \dots \oplus E^{c,c(f)} \oplus E^u$. Moreover, f has a chain of cs -blenders of index $(\dim(E^s), \dots, \dim(E^s) + c(f) - 1)$.

Proof: Take $f \in \mathcal{O}$, then f has hyperbolic periodic points with index $\dim(E^s)$ and $\dim(E^s) + 1$. By Proposition 3.16, there is an open set $\mathcal{U}_1 \subset \mathcal{O}$ arbitrarily close to f , such that any $g \in \mathcal{U}_1$ has a cs -blender of index $\dim(E^s)$ associated to $(q_{1,g}, p_{1,g})$, where $q_{1,g}$ and $p_{1,g}$ denote the continuation

of q_1 and p_1 for g . Note that \mathcal{O} is an open set. Fix a diffeomorphism $f_1 \in \mathcal{U}_1$ and by the same argument, we can find another open set $\mathcal{U}_2 \subset \mathcal{U}_1$ such that any $g \in \mathcal{U}_1$ has a cs -blender of index $\dim(E^s)$ associated to $(q_{2,g}, p_{2,g})$. Inductively, we obtain open sets $\mathcal{U}_{c(f)} \subset \mathcal{U}_{c(f)-1} \subset \dots \mathcal{U}_1$ such that for any $1 \leq i \leq c(f)$ and any $g \in \mathcal{U}_i$, g has a cs -blender of index $\dim(E^s) + i - 1$ associated to $(q_{i,g}, p_{i,g})$.

Take $g \in \mathcal{U}_{c(f)}$, by Lemma 3.2 and using Lemma 3.18 twice, we get a diffeomorphism $g_1 \in \mathcal{U}_{c(f)}$ which is arbitrarily close to g , such that p_{1,g_1} and q_{2,g_1} are homoclinically related to each other. Do such perturbation $c(f) - 1$ times, we obtain a diffeomorphism $g_{c(f)-1} \in \mathcal{U}_{c(f)}$ such that for any $1 \leq i \leq c(f) - 1$, one has that $p_{i,g_{c(f)-1}}$ and $q_{i+1,g_{c(f)-1}}$ are homoclinically related.

Let \mathcal{O}_1 be the subset consisting of all the diffeomorphisms in \mathcal{O} with chains of cs -blenders of index $(\dim(E^s), \dots, \dim(E^s) + c(f) - 1)$. Then we verified the density of \mathcal{O}_1 . Since homoclinic relation is an open property, it must be an open and dense set as stated in the theorem. \square

Proposition 3.20. *Suppose f admits a dominated splitting $E^s \oplus E^{c1} \oplus \dots \oplus E^{cc} \oplus E^u$ with $\dim(E^{ci}) = 1$ for any $1 \leq i \leq c$, and has a chain of blenders $\{(q_i, p_i)\}_{i=1}^{c(f)}$ of index $(\dim(E^s), \dots, \dim(E^s) + c - 1)$. Then for any $(d - (\dim(E^s) + j))$ -dimension disk D which is tangent to a cone field around $E^{c,j+1} \oplus \dots \oplus E^{c,c(f)} \oplus E^u$ and satisfies $D \pitchfork W^s(p_{c(f)}) \neq \emptyset$, $1 \leq j \leq c(f)$, it holds*

$$D \pitchfork W^s(q_{j+1}) \neq \emptyset \quad \text{and} \quad D \pitchfork W^s(p_j) \neq \emptyset.$$

Proof: It suffices to show that, there is $n > 0$ such that

$$f^n(D) \pitchfork W^s(q_{j+1}) \neq \emptyset \quad \text{and} \quad f^n(D) \pitchfork W^s(p_j) \neq \emptyset.$$

We will only prove the first part, since the second part comes analogously.

For any cone field around $E^{c,j+1} \oplus \dots \oplus E^{c,c(f)} \oplus E^u$, note that Df preserves this cone field and contracts its area uniformly. Thus, if D is tangent to a bundle which is tangent to such a cone field, then so do $f^n(D)$.

We prove this by induction. Since $D \pitchfork W^s(p_{c(f)}) \neq \emptyset$, there exists a submanifold D_0 in D with dimension $\dim(E^u) + 1$, which is tangent to a cone field around $E^{c,c(f)} \oplus E^u$ and satisfies $D_0 \pitchfork W^s(p_{c(f)}) \neq \emptyset$. By Proposition 3.16, we have

$$D_0 \pitchfork W^s(q_{c(f)}) \neq \emptyset.$$

Thus by λ -lemma, there is an integer $n_0 > 0$, such that $f^{n_0}(D_0) \pitchfork W^s(p_{c(f)-1}) \neq \emptyset$, since $q_{c(f)}$ and $p_{c(f)-1}$ are homoclinically related. Then $f^{n_0}(D) \pitchfork W^s(p_{c(f)-1}) \neq \emptyset$. There is a submanifold $D_1 \subset D$ with dimension $\dim(E^u) + 2$, such that D_1 is tangent to a cone around $E^{c,c(f)-1} \oplus E^{c,c(f)} \oplus E^u$, and $D_1 \pitchfork W^s(p_{c(f)-1}) \neq \emptyset$. Continue the argument and then we can complete the proof. \square

• New Criteria of ergodicity

Let p be a saddle of a $C^{1+\alpha}$ diffeomorphism f and $Orb(p)$ denote the orbit of p . We write

$$B^s(p, f) = \{x : W^s(Orb(p)) \pitchfork W^u(x) \neq \emptyset\},$$

$$B^u(p, f) = \{x : W^u(Orb(p)) \pitchfork W^s(x) \neq \emptyset\}.$$

The following proposition is a direct corollary of Theorem A of [23] which presents a new criteria of ergodicity.

Proposition 3.21. *Let $f : M \rightarrow M$ be a $C^{1+\alpha}$ diffeomorphism preserving the volume measure. Suppose that p is a saddle of f such that*

$$\omega(B^s(p, f)), \omega(B^u(p, f)) > 0 \quad \text{and} \quad \omega(M \setminus (B^s(p, f) \cup B^u(p, f))) = 0.$$

Then f is ergodic.

Together with the following result of Avila, we will prove Lemma 3.10 and then complete the proof of Theorem 1.3.

Lemma 3.22. *(Theorem 1 in [6]) C^∞ diffeomorphisms are dense in $\text{Diff}_\omega^1(M)$.*

Proof of Lemma 3.10 Recall that the set $\mathcal{O}_1 \subset \mathcal{O}$ is given in Proposition 3.19. We will use Proposition 3.21 to prove that any $C^{1+\alpha}$ volume preserving diffeomorphism $f \in \mathcal{O}_1$ is ergodic. Since C^∞ volume preserving diffeomorphisms are dense in the C^1 topology by Lemma 3.22, we then can conclude that ergodic diffeomorphisms are dense in \mathcal{O}_1 .

Combining Proposition 3.9 and Proposition 3.19, we know that $f \in \mathcal{O}_1$ satisfies the following properties.

- (G1) f is partially hyperbolic, it admits a partially hyperbolic splitting admits a finest dominated splitting $E^s \oplus E^{c1} \oplus \dots \oplus E^{c, c(f)} \oplus E^u$, where $\dim(E^{ci}) = 1$.
- (G2') f has a chain of cs -blenders of index $(\dim(E^s), \dots, \dim(E^s) + c(f) - 1)$ and a chain of cu -blenders of index $(\dim(E^s) + 1, \dots, \dim(E^s) + c(f))$.
- (G3) f is accessible.
- (G4) There is $0 \leq j(f) \leq c(f)$, such that for any integer $1 \leq j \leq j(f)$, one has

$$\int \log \|Df|_{E^{c, j}(x)}\| d\omega(x) < 0,$$

and for any integer $j(f) + 1 \leq j \leq c(f)$, we has

$$\int \log \|Df|_{E^{c, j}(x)}\| d\omega(x) > 0.$$

When $j(f) = 0$ (resp. $j(f) = c(f)$), we take $E^{c, j(f)}(x)$ (resp. $E^{c, j(f)+1}(x)$) to be vanished.

At first, we consider two special cases: $j(f) = 0$ and $j(f) = c(f)$. One can show the ergodicity of f by a theorem of Burns-Dolgopyat-Pesin [18].

Lemma 3.23 (Theorem 4 in [18]). *Let f be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism preserving a smooth measure. Assume that f is accessible and has negative center Lyapunov exponents on a set of positive measure. Then f is stably ergodic.*

From now on, we assume that $0 < j(f) < c(f)$. Denote $\lambda_i^c(x)$ ($1 \leq i \leq c(f)$) to be the center Lyapunov exponent of x along the center bundle E^{ci} . And denote

$$\Lambda^{cs} = \{x | \lambda_{j(f)}^c(x) < 0\} \quad \text{and} \quad \Lambda^{cu} = \{x | \lambda_{j(f)+1}^c(x) > 0\}.$$

Because

$$\int \log \|Df|_{E^{c, j(f)}(x)}\| d\omega(x) < 0 \quad \text{and} \quad \int \log \|Df|_{E^{c, j(f)+1}(x)}\| d\omega(x) > 0,$$

one has $\omega(\Lambda^{cs}) > 0$, $\omega(\Lambda^{cu}) > 0$. Moreover, for $\omega - a.e.$ x , note that the subbundle $E^{c,j(f)}(x)$ is dominated by the other bundle $E^{c,j(f)+1}(x)$ and hence at least one of the Lyapunov exponents $\lambda_{j(f)}^c(x)$ and $\lambda_{j(f)+1}^c(x)$ should be nonzero. Thus we have that $\Lambda^{cs} \cup \Lambda^{cu}$ has full volume.

Recall that f has a chain of cs -blenders of index $(\dim(E^s), \dots, \dim(E^s) + c(f) - 1)$ by $(G2')$, denoted by $\{q_i, p_i\}_{i=1}^{c(f)}$. Then for $i = j(f)$, f has a cs -blender of index $\dim(E^s) + j(f) - 1$ associated to $(q_{j(f)}, p_{j(f)})$. By Proposition 3.21, Lemma 3.10 is a corollary of the following facts: ω -almost every point of Λ^{cu} belongs to $B^s(p_{j(f)}, f)$ and ω -almost every point of Λ^{cs} belongs to $B^u(p_{j(f)}, f)$. We only prove the first part and the proof of the second part is similar.

By Proposition 3.13, ω -almost every point x in Λ^{cu} has local unstable manifold $W_{loc}^u(x)$ of dimension $d - (\dim(E^{cs}) + j(f))$ and moreover, $W_{loc}^u(x)$ is tangent to the bundle $E^{c,j(f)+1} \oplus \dots \oplus E^{c,c(f)} \oplus E^u$.

Note that f has a cs -blender of index $\dim(E^s) + c(f) - 1$ associated to $(q_{c(f)}, p_{c(f)})$. Because f is accessible, the orbit of almost every point is dense in the ambient manifold M . We can assume that the point x is arbitrarily close to the periodic points $p_{c(f)}$ of index $\dim(E^s) + c(f)$. Thus the strong unstable leaf at x should intersect the stable manifold at $p_{c(f)}$ transversely, i.e., $W^u(x) \pitchfork W^s(p_{c(f)}) \neq \emptyset$. Since $x \in \Lambda^{cu}$, by Remark 3.14, we have that $W^u(x) \subset W_{loc}^u(x)$. Then $W_{loc}^u(x) \pitchfork W^s(p_{c(f)}) \neq \emptyset$. Now it follows from Proposition 3.20 that $W_{loc}^u(x) \pitchfork W^s(p_{j(f)}) \neq \emptyset$ by taking $D = W_{loc}^u(x)$. Hence $x \in B^s(p_{j(f)})$ and we conclude the proof. \square

Proof of Theorem 1.3

Since ergodic diffeomorphisms form a G_δ set, Lemma 3.10 implies that there is a generic subset \mathcal{E} among the set of volume preserving diffeomorphisms far from tangencies, such that any diffeomorphism $f \in \mathcal{E}$ is ergodic.

We take $\mathcal{R} = \mathcal{O} \cap \mathcal{E}$ and, together with Proposition 3.9, conclude the proof. \square

3.4 The proof of Theorem 1.5

Before proving Theorem 1.5, we need the following lemma:

Lemma 3.24. *(Theorem 0.1 in [19]) Let f be C^2 , volume-preserving, partially hyperbolic and center bunched. If f is essentially accessible, then f is ergodic, and in fact has the Kolmogorov property.*

Proof of Theorem 1.5: For $\forall f \in \mathcal{CPH}_\omega^1(M)$, by Lemma 2.3, there is a stably accessible perturbation $f_1 \in \mathcal{CPH}_\omega^1(M)$. Applying Lemma 3.22, we can obtain another perturbation f_2 which is C^2 and accessible. Using Lemma 3.24, f_2 is ergodic. Hence the subset consisting of all the ergodic diffeomorphisms is C^1 dense in $\overline{\mathcal{CPH}_\omega^1(M)}$. Since it is well-known that this subset is a G_δ set, we denote its intersection with $\text{int}(\overline{HT})$ and the residual set determined in Lemma 3.1 by \mathcal{R} as the residual set we need.

For any $f \in \mathbb{R}$, in the finest dominated splitting, there are center bundle with dimension larger than 1 by Lemma 3.7. Then the center exponents along this bundle equal, for, otherwise, suppose the exponents are different, then by Lemma 3.1, for almost every point, along the orbit, there is a dominated splitting. Because f is ergodic, this dominated splitting can be extended on the whole manifold. This contradicts our assumption that this splitting is a finest dominated splitting. \square

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